Lecture 3A

Fundamentals of Radiative Transfer in the Earth’s Atmosphere

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Outline

1. Optical Parameters
2. Gaseous Absorption Spectrum
3. Radiative Transfer Equation
4. Discrete-ordinate Solution
5. Emission Approach
6. Scattering Approach
Atmospheric Optical Parameters

Transmittance:

\[ \Upsilon_\nu (z) = \exp\left(-\frac{\tau_\nu}{\mu}\right) = \exp\left(-\int_{z}^{TOA} \frac{K_\nu}{\mu} \, dz\right) \]

Optical thickness or Opacity:

\[ \tau_\nu = \int_{z}^{TOA} K_\nu \, dz \]

\( K_\nu \) is gaseous absorption coefficient
Planck Function

General Expression:

\[ B_\nu(T) = \frac{2hc^2\nu^3}{hc\nu e^{kT} - 1} \]

*Unit:* Energy/time/area/steradian/frequency-interval

Converting to *Brightness Temperature*:

*Temperature* that the Planck Function is equal to measured radiance at a given frequency
Planck Function in Microwave Wavelength

**General Expression:**

\[
B_\nu(T) = \frac{2hc^2\nu^3}{\hbar\nu e^{\frac{h\nu}{kT}} - 1}
\]

*Unit:* Energy/time/area/steradian/frequency-interval

In microwave region: \( \frac{h\nu}{kT} < 1 \)

\[
B_\nu(T) \approx CT
\]

(Rayleigh Jeans Approximation)
What are Measured from Satellites?

Planck Function

This is measured from space

Brightness Temperature

This is typically shown
Instrument Spectrum Allocations

Frequency (GHz)

Pressure 100 hPa

Pressure 500 hPa

Pressure 1013 hPa
MW Stratosphere and Mesosphere Sounding
Millimeter Wavelength Spectroscopy

Pressure 100 hPa

Pressure 300 hPa

Pressure 500 hPa

Transmittance

Frequency (GHz)
Microwave Penetration Depth
Particle Scattering Parameters

Scattering coefficient:

\[ \sigma_s = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n + 1)(|a_n|^2 + |b_n|^2) \]

Extinction coefficient:

\[ \sigma_e = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n + 1)Re(a_n + b_n) \]
Particle Scattering Parameters

Particle Scattering Phase Matrix:

\[ P = \frac{\lambda^2}{\pi \sigma_s} \mathbf{P} \]

\[ \mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{11} & 0 & 0 \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & -P_{34} & P_{33} \end{bmatrix} \]
Radiative Transfer Equation

\[ \mu \frac{dI(\tau, \mu, \phi)}{d\tau} = -I(\tau, \mu, \phi) + J(\tau, \mu, \phi) + S(\tau, \mu, \phi, \mu_0, \phi_0) \]

\[ J = \frac{\omega(\tau)}{4\pi} \int_0^{2\pi} \int_{-1}^{1} M(\tau, \mu, \phi; \mu', \phi') I(\tau, \mu', \phi') d\mu' d\phi' \]

1. Extinction
2. Multiple scattering
3. First scattering & thermal emission
Source Term

\[ S(\tau, \mu, \phi, \mu_0, \phi_0) = (1 - \omega) B \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\omega F_0}{4\pi} \exp\left(-\frac{\tau}{\mu_0}\right) \begin{pmatrix} M_{11}(\phi, \mu_0, \phi_0) \\ M_{12}(\phi, \mu_0, \phi_0) \\ M_{13}(\phi, \mu_0, \phi_0) \\ M_{14}(\phi, \mu_0, \phi_0) \end{pmatrix} \]
Radiative Transfer Coordinate

\[ I_0 = (\beta_0=40.00, \psi_0=100.00) \quad I_S = (\beta_S=45.00, \psi_S=15.00) \]

\[ \Delta \theta = 59.89 \]

Zenith

\[ \beta_0 \quad \psi_0 \quad \psi=90 \quad \Delta \psi \quad \psi_S \quad \beta_S \quad \psi=0 \]
Solution to General Radiative Transfer Problem

Step 1: Fourier harmonic expansion in azimuthal direction

\[
I(\tau, \mu, \phi) = \sum_{m=0}^{2N-1} \left[ I_m^c(\tau, \mu) \cos m(\phi_0 - \phi) + I_m^s(\tau, \mu) \sin m(\phi_0 - \phi) \right]
\]

\[
S(\tau, \mu, \phi) = \sum_{m=0}^{2N-1} \left[ S_m^c(\tau, \mu) \cos m(\phi_0 - \phi) + S_m^s(\tau, \mu) \sin m(\phi_0 - \phi) \right]
\]

\[
\mu \frac{dI_m^c(\tau, \mu)}{d\tau} = I_m^c(\tau, \mu) - S_m^c(\tau, \mu)
\]

\[
- \frac{\omega(\tau)}{4} \int_{-1}^{1} \left[ (1 + \delta_{0m}) M_m^c I_m^c - (1 - \delta_{0m}) M_m^s I_m^s \right] d\mu'
\]

\[
\mu \frac{dI_m^s(\tau, \mu)}{d\tau} = I_m^s(\tau, \mu) - S_m^s(\tau, \mu)
\]

\[
- \frac{\omega(\tau)}{4} \int_{-1}^{1} \left[ (1 - \delta_{0m}) M_m^c I_m^s + (1 - \delta_{0m}) M_m^s I_m^c \right] d\mu'
\]
Solution to General Radiative Transfer Problem

Step 2: Legendre Polynomial expansion in zenith angle direction

\[
\mu \frac{dI_m^c(\tau, \mu_i)}{d\tau} = I_m^c(\tau, \mu) - \frac{\omega(\tau)}{4} \sum_{j=-N}^{N} \omega_j [(1 + \delta_{0m})M_{m}^c(\tau, \mu_i, \mu_j)I_m^c(\tau, \mu_j) - (1 - \delta_{0m})M_{m}^s(\tau, \mu_i, \mu_j)I_m^s(\tau, \mu_j)] - Q_m^c(\tau, \mu_i)
\]

\[
\mu \frac{dI_m^s(\tau, \mu_i)}{d\tau} = I_m^s(\tau, \mu) - \frac{\omega(\tau)}{4} \sum_{j=-N}^{N} \omega_j [(1 - \delta_{0m})M_{m}^c(\tau, \mu_i, \mu_j)I_m^s(\tau, \mu_j) - (1 - \delta_{0m})M_{m}^s(\tau, \mu_i, \mu_j)I_m^c(\tau, \mu_j)] - Q_m^s(\tau, \mu_i)
\]

\[i = \pm 1, \ldots \pm N, m = 0, \ldots, (2N - 1)\]
Step 3: Find general and specific solution

\[ I = \exp[A(\tau - \tau_{l-1})]C_1 + S_1 \]

\[ S_1 = \delta_{m0} \{ B(\tau_{l-1}) \Xi + \frac{B(\tau_{l-1}) - B(\tau_l)}{\tau_{l-1} - \tau_l} \left[ A_l^{-1} \Xi + (\tau - \tau_{l-1}) \Xi \right] \]

\[ + \mu_0 [\mu_0 A_l + \mathbf{E}]^{-1} \frac{\omega F_0}{\pi} \exp(-\tau / \mu_0) \Psi \} \]
**Solution to General Radiative Transfer Problem**

**Step 4: Use boundary conditions to determine coefficients**

\[ I_I(0) = I_0 \]

\[ I_I(\tau_{l-1}) = I_{l-1}(\tau_{l-1}) \]

\[ I_L(\tau_L) = \varepsilon B(T_s) + R I_L(\tau_L) \]

\[ + R_0 \frac{F_0}{\pi} \exp(-\tau_L / \mu_0) \Xi \]

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<th>( \tau_L )</th>
<th>( \mu )</th>
<th>( \phi )</th>
<th>( \mu' )</th>
<th>( \phi' )</th>
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<tr>
<td>( T_0 )</td>
<td>( P_0(\mu, \phi, \mu', \phi', \omega_0) )</td>
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<td>( P_l(\mu, \phi, \mu', \phi', \omega_l) )</td>
<td>( I_l(\tau_l, \mu, \phi) )</td>
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Emission-Based Approximation

\[ \mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - B(\tau) \]

\[ I(\tau_0, \mu) = I(\tau_s, \mu) \exp(-\tau_s / \mu) + \]

\[ \int_0^1 r_s(\mu, \mu') d\mu' \int_{\tau_0}^{\tau_s} B(\tau, T) \exp \left[ -\frac{(\tau - \tau_0)}{\mu'} \right] d\tau / \mu + \]

\[ \int_{\tau_s}^{\tau_0} B(\tau, T) \exp \left[ -\frac{(\tau_s - \tau)}{\mu} \right] d\tau / \mu \]
Emission-Based Approximation (cont.)

\[ T_b = \varepsilon T_s \exp\left(-\frac{\tau_s}{\mu}\right) + T_u + (1 - \varepsilon) T_d \exp\left(-\frac{\tau_s}{\mu}\right) \]

\[ T_d = \int_{\tau_0}^{\tau_s} B(\tau, T) \exp\left(-\frac{(\tau - \tau_0)}{\mu}\right) d\tau / \mu \]

\[ T_u = \int_{\tau_s}^{\tau_0} B(\tau, T) \exp\left(-\frac{(\tau_s - \tau)}{\mu}\right) d\tau / \mu \]

\[ T_b = T_s [1 - (1 - \varepsilon) \Upsilon^2] - \Delta T (1 - \Upsilon)[1 + (1 - \varepsilon) \Upsilon] \]

\[ \Delta T = T_s - T_m \quad \Upsilon = \exp\left(-\frac{\tau_s}{\mu}\right) \]
Emission-Based Approximation (cont.)

\[ T_b = \varepsilon T_s \exp\left(-\frac{\tau_s}{\mu}\right) + T_u + (1 - \varepsilon)T_d \exp\left(-\frac{\tau_s}{\mu}\right) \]

- Surface Emission attenuated by atmosphere
- Atmospheric upwelling radiation
- Atmospheric downwelling radiation reflected by surface and then attenuated by atmosphere
Brightness Temperature vs. Atmospheric and Surface Variables

$T_b$ (K) vs. Transmittance ($\tau$):
- $T_s = 300$ K, $\Delta T = 10$ K
- $\epsilon = 1.00$
- $\epsilon = 0.95$
- $\epsilon = 0.90$

$T_b$ (K) vs. Surface-Air Temperature Diff ($\Delta T_s$):
- $T_s = 300$ K, $\tau = 0.9$
- $\epsilon = 1.00$
- $\epsilon = 0.95$
- $\epsilon = 0.90$

$T_b$ (K) vs. Surface Emissivity ($\epsilon$):
- $\tau = 0.9$, $\Delta T = 10$ K
- $T_s = 310$
- $T_s = 300$
- $T_s = 290$

$T_b$ (K) vs. Surface Temperature ($T_s$):
- $\tau = 0.9$, $\Delta T = 10$ K
- $\epsilon = 1.00$
- $\epsilon = 0.95$
- $\epsilon = 0.90$
Scattering Model Using Two-Stream Approximation

\[
\mu \frac{dI_m^c(\tau, \mu_i)}{d\tau} = I_m^c(\tau, \mu) - \frac{\omega(\tau)}{4} \sum_{j=-N}^{N} w_j[(1+\delta_{0m})M_m^c(\tau, \mu_i, \mu_j)I_m^c(\tau, \mu_j) - (1-\delta_{0m})M_m^s(\tau, \mu_i, \mu_j)I_m^s(\tau, \mu_j)] - Q_m^c(\tau, \mu_i)
\]

If we only consider \(m = 0, N = 1\), thus \(J=-1, 1\) (two streams)

\[
\mu \frac{dI(\tau, \mu)}{d\tau} = [1 - \omega(1-b)]I(\tau, \mu) - \omega b I(\tau, -\mu) - (1 - \omega)B
\]

\[
-\mu \frac{dI(\tau, -\mu)}{d\tau} = [1 - \omega(1-b)]I(\tau, -\mu) - \omega b I(\tau, \mu) - (1 - \omega)B
\]
Scattering Model using Two-Stream Approximation (cont.)

For a single layer of scattering medium:

\[
I(\tau, \mu) = \frac{I_0'[\gamma_1 e^{\kappa(\tau-\tau_1)} - \gamma_2 e^{-\kappa(\tau-\tau_1)}] - I_1'[\beta_3 e^{\kappa(\tau-\tau_0)} - \beta_4 e^{-\kappa(\tau-\tau_0)}]}{\beta_1 \gamma_4 e^{-\kappa(\tau_1-\tau_0)} - \beta_2 \gamma_3 e^{\kappa(\tau_1-\tau_0)}} + B
\]

\[
I(\tau, -\mu) = \frac{I_0'[\gamma_4 e^{\kappa(\tau-\tau_1)} - \gamma_3 e^{-\kappa(\tau-\tau_1)}] - I_1'[\beta_2 e^{\kappa(\tau-\tau_0)} - \beta_1 e^{-\kappa(\tau-\tau_0)}]}{\beta_1 \gamma_4 e^{-\kappa(\tau_1-\tau_0)} - \beta_2 \gamma_3 e^{\kappa(\tau_1-\tau_0)}} + B
\]

\[b: \text{a ratio of backward scattering to the total scattering intensity}\]
\[\omega: \text{single scattering albedoo}\]
\[\kappa: \text{eigenvalue and a function of } b \text{ and } \omega\]
\[B: \text{Planck function}\]
Errors of Two-Stream Model

- Zenith angle = 0°
- Zenith angle = 10°
- Zenith angle = 20°
- Zenith angle = 30°
- Zenith angle = 40°
- Zenith angle = 50°

Best performance!
Summary

1. At microwave region, radiance is a linear function of brightness temperature, and radiance and brightness temperature are inter-changeable in radiative transfer equation.

2. Absorption lines in Lorenz shapes from O2 at 50-60 GHz and H2O at 183 GHz provide sounding capability.

3. Radiative transfer equation is an integral and differential equation, and in general, radiance solution can be solved numerically for a vertically stratified emission and scattering atmosphere.

4. In a scattering-free atmosphere, the solution of radiance can be expressed in an analytic form, i.e., the so-called emission-based approach.

5. In a scattering condition, the radiance can be expressed analytically using a two-stream approximation.


